Another example:

Let $g(z) = z^2$. Prove that $g'(\omega)$ exists for all $\omega \in \mathbb{C}$.

Proof: $\forall \omega \in \mathbb{C}$, observe that

$$\frac{g(\omega+h) - g(\omega)}{h} = \frac{(\omega+h)^2 - \omega^2}{h}$$

$$= \frac{\omega^2 + 2\omega h + h^2 - \omega^2}{h}$$

$$= \frac{2\omega h + h^2}{h} = 2\omega + h$$

$$\therefore \; g'(\omega) = \lim_{h\to 0} \frac{g(\omega+h) - g(\omega)}{h} = \lim_{h\to 0} (2\omega + h) = \boxed{2\omega}$$

by the algebraic continuity thm.

polynomial in h
$\Rightarrow$ continuous
$\Rightarrow \lim_{h\to 0} p(h) = p(0)$.

QED.

Notation $f'(z) = \frac{\partial f}{\partial z}(z) = \frac{df}{dz}(z)$.

Algebraic (complex) Derivative Thm.

① If c is a constant and $g(z) = c \; \forall z$, then $g'(z) = 0$.

② If $n \in \mathbb{N}$, $(z^n)' = nz^{n-1}$.

③ If f is complex diff'ble at $g(a)$ and g is $\mathbb{C}$-diff'ble at a, then $f\circ g$ is complex-diff'ble at a, and

$$(f\circ g)'(a) = f'(g(a))\cdot g'(a).$$

④ If f & g are $\mathbb{C}$-diff'ble at a, then fg is $\mathbb{C}$-diff'ble at a, and

$$(fg)'(a) = f'(a)\,g(a) + g'(a)f(a).$$

⑤ If f & g are $\mathbb{C}$-diff'ble at a and $g(a) \neq 0$, then

$$\left(\frac{f}{g}\right)'(a) = \frac{f(a)g'(a) - g(a)f'(a)}{g(a)^2}$$

linearity of $\mathbb{C}$-deriv.

⑥ Corollary of 4: If $c \in \mathbb{C}$, f is $\mathbb{C}$-diff'ble at a, then

$$(cf)'(a) = c\cdot f'(a)$$

⑦ If f & g are $\mathbb{C}$ diff'ble at a, then $(f+g)$ is $\mathbb{C}$-diff'ble at a

$$\Rightarrow (f+g)'(a) = f'(a) + g'(a).$$

Extra-fact: If f is $\mathbb{C}$diff'ble at a, then f is cont. at a.

Proof: $\lim_{h\to 0}\left(f(a+h)-f(a)\right) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)\cdot h$

$\underset{\text{Alg limit thm}}{=} f'(a)\cdot 0 = 0.$

$\therefore \lim_{\substack{h\to 0 \\ z\to a}} f(\underbrace{a+h}_{z}) = \lim_{h\to 0} f(a) = f(a).$

$\therefore$ f is continuous at a.

As a result of ↗ polynomials & rational fcns in z are $\mathbb{C}$-diff'ble on their domains. (Not polynomials in $\bar{z}$!)

(Note: continuous does not imply $\mathbb{C}$-diff'ble. eg. $f(z)=\mathrm{Re}(z)$ is continuous at all z, but never $\mathbb{C}$-diff'ble.)

Example Proof: Let's prove that if f, g are $\mathbb{C}$-diff'ble @ $a\in\mathbb{C}$, then fg is $\mathbb{C}$-diff'ble at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

Proof: Consider $\dfrac{(fg)(a+h)-(fg)(a)}{h}$

$= \dfrac{f(a+h)g(a+h) - f(a)g(a)}{h}$

$$= \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h}$$

$$= \frac{f(a+h)-f(a)}{h}\, g(a+h) + f(a)\, \frac{g(a+h)-g(a)}{h}$$

Note that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$,

$\lim_{h\to 0} g(a+h) = g(a)$ since g is continuous at a (because diff'ble $\Rightarrow$ cont.)

$\lim_{h\to 0} \frac{g(a+h)-g(a)}{h} = g'(a)$

$\therefore$ By the alg. limit thm,

$$\underbrace{\lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}}_{=(fg)'(a).} = f'(a)g(a) + f(a)g'(a).$$

$\blacksquare$

Side Comment: Differentiable functions

$g: \mathbb{R}^2 \to \mathbb{R}^2$. $\quad g(x,y) = (u(x,y), v(x,y))$

Defn. A function $g: \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{R}$-diff'ble at $a = (a_1, a_2) \in \mathbb{R}^2$ if $\exists\, L: \mathbb{R}^2 \to \mathbb{R}^2$ that is a linear map such that

$$\lim_{h=(h_1,h_2)\to(0,0)} \frac{g(a+h) - g(a) - Lh}{|h|} = 0.$$

(where $|h| = \sqrt{h_1^2 + h_2^2}$.)

If g is diff'ble at $a = (a_1, a_2)$, then

$$L = \begin{pmatrix} u_x(a_1,a_2) & u_y(a_1,a_2) \\ v_x(a_1,a_2) & v_y(a_1,a_2) \end{pmatrix}$$

(ie partial derivatives exist.)

derivative matrix of g.

<u>Numerator:</u>

$$g(a+h) - g(a) - Lh$$

$$\begin{pmatrix} u(a_1+h_1, a_2+h_2) \\ v(a_1+h_1, a_2+h_2) \end{pmatrix} - \begin{pmatrix} u(a_1,a_2) \\ v(a_1,a_2) \end{pmatrix} - \overbrace{\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}}^{g'(a)} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= \begin{pmatrix} u(a+h) - u(a) - \nabla u \cdot h \\ v(a+h) - v(a) - \nabla v \cdot h \end{pmatrix}$$ as a vector

---

eg: If $g(x,y) = (x, 0)$ $\quad [g(z) = \mathrm{Re}(z)]$

$u(x,y) = x$

$v(x,y) = 0$, $\quad g' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = L$

<u>Numerator</u> $g(a+h) - g(a) - Lh$

$$= \begin{pmatrix} a_1+h_1 \\ 0 \end{pmatrix} - \begin{pmatrix} a_1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$g(x,y) = (x,0)$

$\Rightarrow \lim_{h\to 0} \dfrac{g(a+h) - g(a) - Lh}{|h|} = \lim_{h\to 0} \dfrac{(0,0)}{|h|} = \boxed{(0,0)}$

$\therefore g$ is $\mathbb{R}$-diff'ble at every $a = (a_1, a_2) \in \mathbb{R}^2$

Nice facts in Real variables: If f is C^1 (partial derivatives are continuous) at a point, then f is $\mathbb{R}$-diff'ble at that point.

What is different when $f: \underset{\mathbb{C}}{\underset{\shortparallel}{\mathbb{R}^2}} \to \underset{\mathbb{C}}{\underset{\shortparallel}{\mathbb{R}^2}}$ is $\mathbb{C}$-differentiable?

$f(a+h) - f(a) \simeq f'(a)h$

$f(a+h) - f(a) = \underbrace{f'(a)h}_{Lh} + o(|h|)$

This goes to 0 faster than $|h|$.

$f'(a) = (w_1 + w_2 i) \in \mathbb{C}$.

$h = h_1 + h_2 i$

$f'(a)h = Lh$

$$= (\omega_1 + \omega_2 i)(h_1 + h_2 i)$$

$$= (\omega_1 h_1 - \omega_2 h_2) + i(\omega_2 h_1 + \omega_1 h_2)$$

Vector form:

$$f'(a)h = Lh = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= \begin{pmatrix} L_{11} h_1 + L_{12} h_2 \\ L_{21} h_1 + L_{22} h_2 \end{pmatrix}$$

$$= (L_{11} h_1 + L_{12} h_2) + i(L_{21} h_1 + L_{22} h_2)$$

$$\Rightarrow \quad L_{11} = \omega_1, \; L_{12} = -\omega_2$$

$$L_{21} = \omega_2, \; L_{22} = \omega_1$$

$\Rightarrow$ For $\mathbb{C}$-differentiable, L can't just be anything, it must have the form

$$L = \begin{pmatrix} \omega_1 & -\omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}.$$

<u>The point is:</u> For $\mathbb{R}$-differentiable as a fcn from $\mathbb{R}^2 \to \mathbb{R}^2$, the derivative matrix can be any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. But for $\mathbb{C}$-differentiable, the derivative matrix must have the form

$$L = \begin{pmatrix} \omega_1 & -\omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}.$$

Corollary of this Calculation:

for $x, y \in \mathbb{R}$

If $f(x+iy) = u(x,y) + i(v(x,y))$, is a function defined on a neighborhood of $(x,y) \in \mathbb{R}^2$
$x+iy \in \mathbb{C}$,

then f is $\mathbb{C}$-differentiable at $x_0 + iy_0$

$\Longrightarrow$ $u_x = v_y$ at (x_0, y_0)
and $u_y = -v_x$

Called the Cauchy-Riemann equations.

Partial Converse. If $f(x+iy) = u(x,y) + i(v(x,y))$ is C^1 in a neighborhood of (x_0+iy_0)

1st partial derivs are continuous, ie u_x, u_y, v_x, v_y are cont. at (x_0, y_0)

and the C-R equns $\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}$ are true at (x_0, y_0), then f is $\mathbb{C}$-diff. at (x_0, y_0).

We say a function $f: \underset{\mathbb{C}}{\underset{\cap \text{t}}{U}} \to \mathbb{C}$

is <u>holomorphic</u> (<u>analytic</u>) at $z_0 \in U$

if $\exists\, \varepsilon > 0$ s.t. $f'(z)$ exists

at every $z \in B(z_0, \varepsilon)$.

(ie $\mathbb{C}$-differentiable near z_0)